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On the Failure of Proximinality of Tensor-Product Subspaces

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An important open problem concerning the approximation of bivariate functions by separable functions is whether the tensor-product subspace,

$$C(S)\otimes H+G\otimes C(T),$$

is proximinal in $C(S\times T)$, when H and G are Haar subspaces of $C(T)$ and $C(S)$, respectively. In the present paper, we prove that, in general, this subspace is not proximinal. © 1990 Academic Press, Inc.

1. INTRODUCTION

In a normed linear space, any element possesses an element of best approximation in any finite-dimensional subspace. This is often not the case if the dimension of the subspace is infinite.

In this paper we consider the linear space $C(S\times T)$ of real-valued continuous functions on the unit square $[-1,1]\times[-1,1]$ endowed with the uniform norm, where $S=T=[-1,1]$.

It has been shown by Diliberto and Straus [3] that the subspace

$$\{x(s)+y(t): x, y\in C[-1,1]\}$$

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is proximinal in $C(S \times T)$. The same is true for the subspaces

$$\left\{x(s) + \sum_{j=1}^{n} s^{j-1} y_j(t) : x, y_1, ..., y_n \in C[-1, 1]\right\}$$

(see Cheney and Respess [2]).

In this paper we shall construc t a function $f \in C(S \times T)$ which does not have an element of best approximation (with respect to the uniform norm) in the subspace

$$W = C(S) \otimes H + G \otimes C(T) \tag{1.1}$$

when H and G are taken to be the 2-dimensional spaces of polynomials of degree 1. In this case, the elements of W have the form

$$w(s, t) = x_0(s) + t x_1(s) + y_0(t) + s y_1(t), \qquad \text{with} \quad x_i \in C(S),\ y_i \in C(T).$$

Earlier, one of the authors [4] has shown that any function f in $C(S \times T)$ has a best approximation in W if the partial derivative $\partial f/\partial s$ exists at the boundary points $(1, t)$, $t \in T$ and $(\partial f/\partial s)(1, \cdot) \in C(T)$.

It is also known (see Cheney and v. Golitschek [1]) that the subspace

$$W_1 = l_\infty(S) \otimes H + G \otimes l_\infty(T)$$

is proximinal in $l_\infty(S \times T)$. Furthermore, if H and G are 2-dimensional spaces of polynomials of degree 1, and f is an element of $C(S \times T)$, it possesses at least one best approximation w in $l_\infty(S) \otimes H + G \otimes l_\infty(T)$ that is continuous on the interior of $S \times T$.

2. Construction of the Function f

We start by defining the function f on the set $\Lambda \times \Lambda$ where $\Lambda = \{\lambda_j\}_{j=0}^{\infty}$ is given by $\lambda_j = 1 - 2^{-j}$, $j = 0, 1, 2, ...$ We set

$$f(0, 0) = 0, \qquad f(\lambda_1, 0) = -3, \qquad f(\lambda_2, 0) = 3,$$

$$f(\lambda_1, \lambda_1) = 3, \qquad f(0, \lambda_1) = 3, \qquad f(0, \lambda_2) = -3,$$

$$f(\lambda_i, 0) = f(0, \lambda_i) = 0, \qquad i \geqslant 3,$$

$$f(\lambda_i, \lambda_j) = (-1)^j (1 - \lambda_i) + (-1)^i (1 - \lambda_j)$$

$$\text{for} \quad j \geqslant 1,\ i = j+1, \text{ and } i = j+2,$$

$$f(\lambda_i, \lambda_j) = (-1)^j (\lambda_j - \lambda_i) \qquad \text{for} \quad j \geqslant 1,\ i \geqslant j+3,$$

$$f(\lambda_i, \lambda_j) = 0 \qquad \text{for} \quad j \geqslant 2,\ 1 \leqslant i \leqslant j.$$

We extend f onto $[0, 1] \times [0, 1]$ as follows.

First, for $j=0, 1, ..., f(\cdot, \lambda_j)$ is linear in each of the intervals $\lambda_i \leqslant s \leqslant \lambda_{i+1}$, $i \geqslant 0$, and continuous on $[0, 1)$. Then, for each $s \in [0, 1)$, $f(s, \cdot)$ is again defined by linear interpolation of the values $f(s, \lambda_j)$, $f(s, \lambda_{j+1})$ in the intervals $\lambda_j \leqslant t \leqslant \lambda_{j+1}$, $j \geqslant 0$.

Finally we set

$$f(0, 1) = f(1, 0) = f(1, 1) = 0,$$

$$f(\lambda_i, 1) = 0 \qquad \text{for} \quad i \geqslant 1,$$

$$f(1, \lambda_j) = (-1)^j (\lambda_j - 1) \qquad \text{for} \quad j \geqslant 1,$$

and again define $f(1, \cdot)$ and $f(\cdot, 1)$ by linear interpolation of the values in the intervals $\lambda_j \leqslant t \leqslant \lambda_{j+1}$, and $\lambda_i \leqslant s \leqslant \lambda_{i+1}$, $j \geqslant 0$, $i \geqslant 0$, respectively.

It is easy to confirm that f is continuous on $[0, 1] \times [0, 1]$, even Lipschitzian, and that $f(s, 0) = -f(0, s)$, $0 \leqslant s \leqslant 1$. Therefore, f can be uniquely extended on the square $[-1, 1] \times [-1, 1]$ such that the identity

$$f(s, t) = -f(t, -s), \qquad (s, t) \in S \times T \tag{2.1}$$

holds. Also we note that (2.1) implies $f(s, t) = f(-s, -t) = -f(-t, s)$. The function f is continuous on $S \times T$, even Lipschitzian.

3. The Approximating Function

Let x and y be the bounded functions on $[-1, 1]$, continuous on $(-1, 1)$, which have the following properties.

$x(0) = x(1) = 0$, $x(\lambda_i) = (-1)^{i+1}$, $i \geqslant 1$,

x is linear on each interval $[\lambda_i, \lambda_{i+1}]$, $i \geqslant 0$,

x is even on $[-1, 1]$,

y is odd on $[-1, 1]$ and $y(t) = -x(t)$ for $0 \leqslant t \leqslant 1$.

We define the approximating function by

$$w(s, t) = x(s) - x(t) + sy(t) + ty(s), \qquad (s, t) \in S \times T. \tag{3.1}$$

Clearly, w has (as f) the property

$$w(s, t) = -w(t, -s), \qquad (s, t) \in S \times T. \tag{2.1$'$}$$

We claim that

$$\|f + w\| \leqslant 2 \tag{3.2}$$

is valid on $[-1, 1] \times [-1, 1]$. Indeed, we have for $F := f + w$

$$\begin{aligned} F(\lambda_1, 0) &= -2, \qquad F(\lambda_2, 0) = 2, \\ F(0, \lambda_1) &= 2, \qquad F(0, \lambda_2) = -2, \end{aligned} \tag{3.3}$$

$$F(\lambda_1, \lambda_1) = 2, \qquad \text{since } \lambda_1 = 1/2 \text{ and } y(\lambda_1) = -1$$

$$F(0, 0) = 0, \qquad F(\lambda_i, 0) = w(\lambda_i, 0) = (-1)^{i+1} \text{ for } i \geqslant 3. \tag{3.4}$$

For $j \geqslant 1$, $i = j + 1$, and $i = j + 2$, one gets

$$w(\lambda_i, \lambda_j) = (1 - \lambda_j)(-1)^{i+1} - (1 + \lambda_i)(-1)^{j+1}$$

and thus

$$F(\lambda_i, \lambda_j) = 2(-1)^j, \qquad i = j + 1, i = j + 2, j \geqslant 1. \tag{3.5}$$

For $j \geqslant 1$, $i \geqslant j + 3$, we have

$$\begin{aligned} |F(\lambda_i, \lambda_j)| &= |(-1)^j (\lambda_j - \lambda_i) + w(\lambda_i, \lambda_j)| \\ &= |(-1)^j (1 + \lambda_j) + (-1)^{i+1} (1 - \lambda_j)| \leqslant 2, \end{aligned}$$

and for $j \geqslant 2$, $1 \leqslant i \leqslant j$,

$$|F(\lambda_i, \lambda_j)| = |w(\lambda_i, \lambda_j)| \leqslant 1 - \lambda_j + 1 + \lambda_i \leqslant 2,$$

where the last inequality follows since $\lambda_i \leqslant \lambda_j$ for $i \leqslant j$. Hence we have proved that $|F(s, t)| \leqslant 2$ on $\Lambda \times \Lambda$.

On the boundary, we have

$$\begin{aligned} F(1, 0) &= F(0, 1) = F(1, 1) = 0, \\ F(\lambda_i, 1) &= w(\lambda_i, 1) = 0, \qquad i \geqslant 1, \\ |F(1, \lambda_j)| &= |(-1)^j (\lambda_j - 1) + w(1, \lambda_j)| \\ &= |(-1)^j (\lambda_j - 1) + 2(-1)^j| \\ &= |1 + \lambda_j| < 2, \qquad j \geqslant 1. \end{aligned}$$

By the definition (3.1), each of the functions $w(\cdot, t)$, $0 \leqslant t \leqslant 1$, is linear in $\lambda_i \leqslant s \leqslant \lambda_{i+1}$, $i \geqslant 0$, and each of the functions $w(s, \cdot)$, $0 \leqslant s \leqslant 1$, is linear in $\lambda_j \leqslant t \leqslant \lambda_{j+1}$, $j \geqslant 0$. Since f has the same property, it follows that

$$|F(s, t)| \leqslant 2 \qquad \text{on} \quad [0, 1] \times [0, 1].$$

Finally, since f and w have the property (2.1), we have even established that on $[-1, 1] \times [-1, 1]$

$$\|f + w\| = 2. \tag{3.6}$$

4. Failure of Proximinality

We will prove in two ways that there does not exist a function $w^* \in W$, such that

$$\|f + w^*\| \leqslant 2. \tag{4.1}$$

Suppose that there exists such a continuous w^*. Since f has the property (2.1), there also exists a continuous function (again called w^*) which satisfies (2.1) and (4.1). Indeed, if w is a continuous function such that $\|f + w\| \leqslant 2$, then define w^* by setting

$$w^*(s, t) = \tfrac{1}{4}[w(s, t) + w(-s, -t) - w(t, -s) - w(-t, s)].$$

It follows that $w^*(s, t) = w^*(-s, -t) = -w^*(t, -s) = -w^*(-t, s)$. Since f also has these properties from (2.1), we see easily that $\|f + w^*\| \leqslant 2$. Let $w(s, t) = x_0(s) + tx_1(s) + y_0(t) + sy_1(t)$, then by definition of w^*,

$$\begin{aligned} w^*(s, t) = \tfrac{1}{4}\{&[x_0(s) + x_0(-s) - y_0(s) - y_0(-s)] \\ &- [x_0(t) + x_0(-t) - y_0(t) - y_0(-t)] \\ &+ s[x_1(t) + y_1(t) - x_1(-t) - y_1(-t)] \\ &+ t[x_1(s) + y_1(s) - x_1(-s) - y_1(-s)]\} \end{aligned}$$

thus if we define

$$\begin{aligned} x^*(s) &= \tfrac{1}{4}[x_0(s) + x_0(-s) - y_0(s) - y_0(-s)], \\ y^*(s) &= \tfrac{1}{4}[x_1(s) + y_1(s) - x_1(-s) - y_1(-s)], \end{aligned}$$

w^* is then of the form

$$w^*(s, t) = x^*(s) - x^*(t) + sy^*(t) + ty^*(s), \qquad -1 \leqslant s, t \leqslant 1 \tag{4.2}$$

and $x^* \in C[-1, 1]$ is even, $y^* \in C[-1, 1]$ is odd, hence $y^*(0) = 0$, and without loss of generality, $x^*(0) = 0$.

There are two ways to show that w^* cannot be continuous at $(s, t) = (1, 1)$.

A. *The First Method of Proof*

Let w be the function in Section 3 and consider the function $z := w - w^*$. Because of (3.1) and (4.2), z is also of the form

$$z(s, t) = u(s) - u(t) + sv(t) + tv(s)$$

with bounded functions u and v.

Let us denote $u_i := u(\lambda_i)$ and $v_j := v(\lambda_j)$. Then we have $u_0 = v_0 = 0$, and (3.3), (3.5), (4.1) imply

$$z(\lambda_1, 0) \leqslant 0$$
$$z(\lambda_2, 0) \geqslant 0$$
$$z(\lambda_1, \lambda_1) \geqslant 0$$
$$(-1)^j z(\lambda_{j+1}, \lambda_j) \geqslant 0, \qquad (-1)^j z(\lambda_{j+2}, \lambda_j) \geqslant 0, \text{ for } j \geqslant 1.$$

These inequalities are equivalent to

$$\begin{aligned} &u_1 \leqslant 0, \\ &u_2 \geqslant 0, \\ &v_1 \geqslant 0, \\ &z(\lambda_2, \lambda_1) = u_2 - u_1 + \lambda_2 v_1 + \lambda_1 v_2 \leqslant 0 \\ &\text{which implies } v_2 \leqslant 0. \end{aligned} \tag{4.3}$$

$$(-1)^j (u_{j+2} - u_j + \lambda_{j+2} v_j + \lambda_j v_{j+2}) \geqslant 0, \qquad j \geqslant 1, \tag{4.4}$$

$$(-1)^{j+1} (u_{j+2} - u_{j+1} + \lambda_{j+2} v_{j+1} + \lambda_{j+1} v_{j+2}) \geqslant 0, \qquad j \geqslant 1. \tag{4.5}$$

The sum of (4.4) and (4.5) is

$$(-1)^j (u_{j+1} - u_j + \lambda_{j+2} v_j - \lambda_{j+2} v_{j+1} + [\lambda_j - \lambda_{j+1}] v_{j+2}) \geqslant 0,$$

which implies the inequalities

$$(-1)^j v_{j+2} \leqslant \frac{(-1)^j}{\lambda_{j+1} - \lambda_j} (u_{j+1} - u_j + \lambda_{j+2} v_j - \lambda_{j+2} v_{j+1}), \qquad j \geqslant 1. \tag{4.6}$$

The inequality (4.4) implies

$$(-1)^j u_{j+2} \geqslant (-1)^j (u_j - \lambda_{j+2} v_j - \lambda_j v_{j+2}), \qquad j \geqslant 1. \tag{4.7}$$

It is now easy to show that all u_j and v_j have to vanish: By (4.3), (4.6), (4.7) it follows (by induction) that $(-1)^j v_j \leqslant 0$, $(-1)^j u_j \geqslant 0$, for all $j \geqslant 1$. Hence (4.6) and (4.7) imply that $(-1)^j u_j \to \infty$, $(-1)^j v_j \to -\infty$, as $j \to \infty$ if at least one of the v_i or u_j is non-zero.

Since all $u_j = 0$, $v_j = 0$ it follows that the functions x and x^*, y and y^* are identical on the subset $\lambda = \{\lambda_j\}_{j=0}^{\infty}$ which has a cluster point at 1. Hence x^* and y^* are discontinuous at 1.

B. *The Second Method of Proof*

The second method is based on the following

LEMMA. *There exists a continuous linear functional*

$$\Phi: C([0,1]\times[0,1]) \to \mathscr{R}$$

of the form

$$\Phi = -c_1\, g(\lambda_1, \lambda_0) + c_1\, g(\lambda_2, \lambda_0) + c_2\, g(\lambda_1, \lambda_1) + \sum_{j=1}^{\infty} (-1)^j \left(c_{2j+1}\, g(\lambda_{j+1}, \lambda_j) + c_{2j+2}\, g(\lambda_{j+2}, \lambda_j)\right) \tag{4.8}$$

with positive coefficients c_j, $j \geqslant 1$, and $\sum_{j=1}^{\infty} c_j < \infty$ which annihilates the subspace

$$W_0 = \{w : w \text{ is a function of the form (3.1) with bounded } x, y\}.$$

Proof. Let Φ be of the form (4.8) with positive c_i and $\sum c_j < \infty$. Φ annihilates W_0 if and only if Φ annihilates any function $w \in W_0$ of the forms

$$x_i(s) - x_i(t), \qquad s x_i(t) + t x_i(s),\ i \geqslant 1, \tag{4.9}$$

where

$$x_i(\lambda_k) = \begin{cases} 1, & k = i; \\ 0, & k \neq i,\ k \geqslant 0. \end{cases}$$

The identities $\Phi(w) = 0$ for the functions w in (4.9) are equivalent to the infinite system of linear equations

$$\begin{aligned}
-c_1 + c_3 + c_4 &= 0 \\
c_1 - c_3 - c_5 - c_6 &= 0 \\
-c_{2i-1} + c_{2i-2} - c_{2i+1} - c_{2i+2} &= 0, \qquad i \geqslant 3 \\
2\lambda_1 c_2 - \lambda_2 c_3 - \lambda_3 c_4 &= 0 \\
-\lambda_1 c_3 + \lambda_3 c_5 + \lambda_4 c_6 &= 0 \\
-\lambda_{i-1} c_{2i-1} + \lambda_{i-2} c_{2i-2} + \lambda_{i+1} c_{2i+1} + \lambda_{i+2} c_{2i+2} &= 0, \qquad i \geqslant 3,
\end{aligned}$$

which is equivalent

$$
\begin{aligned}
c_1 &= c_3 + c_4 \\
c_2 &= \lambda_2 c_3 + \lambda_3 c_4 \qquad (\text{since } \lambda_1 = \tfrac{1}{2}) \\
c_3 &= \frac{1}{\lambda_1} (\lambda_3 c_5 + \lambda_4 c_6) \\
c_{2i} &= c_{2i+1} + c_{2i+2}, \qquad i \geq 2 \\
c_{2i+1} &= \frac{1}{\lambda_i - \lambda_{i-1}} (\lambda_{i-1} c_{2i+2} + \lambda_{i+2} c_{2i+3} + \lambda_{i+3} c_{2i+4}), \qquad i \geq 2.
\end{aligned}
\tag{4.10}
$$

We now show that (4.10) has a positive solution $\{c_i\}_{i=1}^{\infty}$. For any integer $N \geq 2$ there exists a unique positive finite sequence $\{c_v^{(N)}\}_{v=1}^{2N+4}$ which satisfies

$$c_{2N+2}^{(N)} = c_{2N+3}^{(N)} = c_{2N+4}^{(N)} > 0, \qquad c_1^{(N)} = 1$$

and the identities in (4.10) for $i \leq N$.

By Cantor's diagonalization process we find a positive sequence $\{c_i\}_{i=1}^{\infty}$ with $c_0 = 1$ which satisfies (4.10) for all i. Clearly, $\sum_{i=1}^{\infty} c_i < \infty$ since $\lambda_i - \lambda_{i-1} = 2^{-i}$ and $\lambda_i \to 1$,

This completes the proof of our lemma. ∎

We use now our lemma to show that the function w^* is not continuous. Since $\Phi(w) = 0$ for the function w in Section 3, and $\Phi(w^*) = 0$ we get by (3.3) and (3.5)

$$
\begin{aligned}
\Phi(f) &= \Phi(f + w) = \Phi(F) \\
&= 2c_1 + 2c_1 + 2c_2 + 2 \sum_{j=3}^{\infty} c_j.
\end{aligned}
$$

On the other hand,

$$\Phi(f) = \Phi(f + w^*)$$

which is valid if and only if $f + w$ and $f + w^*$ and thus w and w^* are identical on the support of Φ, i.e., on

$$\{(\lambda_1, \lambda_0), (\lambda_2, \lambda_0), (\lambda_1, \lambda_1), (\lambda_{j+1}, \lambda_j), (\lambda_{j+2}, \lambda_j)\ j \geq 1\}.$$

But this implies that w^* is (like w) discontinuous at point (1, 1).

5. REMARKS

Remark 1. In [1], M. v. Golitschek and E. W. Cheney prove that if G and H are 2-dimensional Haar subspaces containing the constants in $C(S)$ and $C(T)$, respectively, then each element f of $C(S \times T)$ has a best approximation in W_1 which is continuous on the interior of $S \times T$. But this is not true in the general case. Let $T1 = [0, 1]$. We can show that

THEOREM 1. *There exist $\bar{G}$ and H, that are 2-dimensional Haar subspaces in $C(S)$ andf $C(T1)$, respectively, such that there is an element f of $C(S \times T1)$ which has no best approximation in $\bar{W}_1$ which is continuous on the interior of $S \times T1$.*

We need two lemmas for proving the result. These are elementary and are given without proofs. Let $H = \{1, t\}$, $G = \{1, s\}$, and let $\bar{G} = \operatorname{span}\{g_1, g_2\}$, where

$$g_1(s) = \begin{cases} 1, & \text{for } s > 0; \\ 1 + s, & \text{for } s < 0, \end{cases} \qquad g_2(s) = \begin{cases} s, & \text{for } s > 0; \\ s/2, & \text{for } s < 0. \end{cases}$$

LEMMA 1. *The $\bar{G}$ defined above is a Haar subspace of $C(S)$.*

By applying the above result to the domain $[0, 1] \times [0, 1]$, we infer that there is a continuous function f_0 on $[0, 1] \times [0, 1]$ that has no best approximation in $\Pi_1[0, 1] \otimes C[0, 1] + C[0, 1] \otimes \Pi_1[0, 1]$. Let

$$f(s, t) = \begin{cases} f_0 & \text{for } (s, t) \in [0, 1] \times [0, 1]; \\ (1 + \frac{1}{2}s) f_0(0, t) + \frac{1}{2}s f_0(1, t) & \text{for } (s, t) \in [-1, 0] \times [0, 1]. \end{cases}$$

Clearly f is an element of $C(S \times T1)$.

Let $W_1 = l_\infty([0, 1]) \otimes H + G \oplus l_\infty([0, 1])$, and let $\bar{W}_1 = l_\infty(S) \otimes H + \bar{G} \otimes l_\infty([0, 1])$.

LEMMA 2. *Let f, H, and $\bar{G}$ be defined above. Then the following equality holds,*

$$\operatorname{dist}(f_0, W_1) = \operatorname{dist}(f, \bar{W}_1).$$

Proof of Theorem 1. We shall prove that the function f defined above has no best approximation in $\bar{W}_1$ that is continuous on the interior of $S \times T1$. In fact, if f has a best approximation in $\bar{W}_1$ that is continuous in the interior of $S \times T1$, then f_0 has a best approximation in W_1 that is continuous in $[0, 1) \times (0, 1)$. This is just the case 2b of the proof of Theorem in [4]. Thus we conclude that f_0 has a best approximation in W. This contradicts the choice of f_0 that has no best approximation in W. ∎

Remark 2. A first counterexample for the failure of proximinality of the tensor-product space (1.1) was submitted by the first author in Spring 1987 using the method B. The second author simplified it and added the method A.

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References

1. M. v. Golitschek and E. W. Cheney, The best approximation of bivariate functions by separable functions, *Contemp. Math.* **21** (1983), 125–136.
2. J. R. Respess and E. W. Cheney, Best approximation problems in tensor product spaces, *Pacific J. Math.* **102** (1982), 437–446.
3. S. P. Diliberto and E. G. Strauss, On the approximation of a function of several variables by the sum of functions of fewer variables, *Pacific J. Math.* **1** (1951), 195–210.
4. M. v. Golitschek, On the existence of continuous best approximations in tensor-product subspaces of univariate functions, *in* "Approximation Theory IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 475–481, Academic Press, New York, 1983.